Some useful Gaussian and matrix equations

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Appendix: Some useful Gaussian identities

If x is multivariate Gaussian with mean μ and covariance matrix Σ

$$p(\mathbf{x};\mu,\Sigma) \;=\; (2\pi|\Sigma|)^{-D/2}\exp\big(-(\mathbf{x}-\mu)^\top\Sigma^{-1}(\mathbf{x}-\mu)/2\big),$$

then

$$\begin{aligned}\mathbb{E}[\mathbf{x}] &= \mu,\\ \mathbb{V}[\mathbf{x}] &= \mathbb{E}[(\mathbf{x}-\mathbb{E}[\mathbf{x}])^2] \;=\; \Sigma.\end{aligned}$$

For any matrix A, if $\mathbf{z} = A\mathbf{x}$ then $\mathbf{z}$ is Gaussian and

$$\begin{aligned}\mathbb{E}[\mathbf{z}] &= A\mu,\\ \mathbb{V}[\mathbf{z}] &= A\Sigma A^\top.\end{aligned}$$

Matrix and Gaussian identities cheat sheet

Matrix identities

- Matrix inversion lemma (Woodbury, Sherman & Morrison formula)

$$(Z + UWV^\top)^{-1} = Z^{-1} - Z^{-1}U(W^{-1} + V^\top Z^{-1}U)^{-1}V^\top Z^{-1}$$

- A similar equation exists for determinants

$$|Z + UWV^\top| = |Z|\,|W|\,|W^{-1} + V^\top Z^{-1}U|$$

The product of two Gaussian density functions

$$\mathcal{N}(\mathbf{x}|\mathbf{a}, A)\,\mathcal{N}(P^\top \mathbf{x}|\mathbf{b}, B) = z_c\,\mathcal{N}(\mathbf{x}|\mathbf{c}, C)$$

- is proportional to a Gaussian density function with covariance and mean

$$C = \left(A^{-1} + P\,B^{-1}P^\top\right)^{-1} \qquad\qquad c = C\,\left(A^{-1}\mathbf{a} + P\,B^{-1}\,\mathbf{b}\right)$$

- and has a normalizing constant z_c that is Gaussian both in $\mathbf{a}$ and in $\mathbf{b}$

$$z_c = (2\,\pi)^{-\frac{m}{2}}|B + P^\top A\,P|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{b} - P^\top\,\mathbf{a})^\top \left(B + P^\top A\,P\right)^{-1}(\mathbf{b} - P^\top\,\mathbf{a})\right)$$